

Towards the construction of $N=2$ supersymmetric integrable hierarchies

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Abstract

We formulate a conjecture for the three different Lax operators that describe the bosonic sectors of the three possible $N = 2$ supersymmetric integrable hierarchies with $N = 2$ super W_n second hamiltonian structure. We check this conjecture in the simplest cases, then we verify it in general in one of the three possible supersymmetric extensions. To this end we construct the $N = 2$ supersymmetric extensions of the Generalized Non-Linear Schrödinger hierarchy by exhibiting the corresponding super Lax operator. To find the correct hamiltonians we are led to a new definition of super-residues for degenerate $N=2$ supersymmetric pseudodifferential operators. We have found a new non-polynomial Miura-like realization for $N = 2$ superconformal algebra in terms of two bosonic chiral–anti–chiral free superfields.

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1 Introduction

The construction of $N = 2$ supersymmetric integrable hierarchies is in progress. The motivations for studying them are diverse. On the one hand we have quite a good acquaintance with purely bosonic hierarchies and their connection with physical models (2D and topological field theories), while our knowledge of $N = 2$ supersymmetric integrable hierarchies is still scanty; in this regard the situation is quite different from conformal field theories, where a great deal of attention has been paid to the $N = 2$ supersymmetric extensions. On the other hand, once our understanding of $N = 2$ supersymmetric integrable hierarchies becomes satisfactory, we may hope to find a link with physical models, e.g. with untwisted $N = 2$ conformal field theories, in the wake of the relation between bosonic integrable hierarchies and topological field theories. A motivation of a different kind is the mathematical problem itself, which is challenging and not devoid of surprises, such as, for example, the existence of three different $N = 2$ extensions.

In this letter we present a conjecture, and support it with many examples, for the $N = 2$ supersymmetric extensions of bosonic hierarchies based on W_n algebras. This conjecture is based on results obtained in [1, 2, 3] concerning the so-called (n, m) -KdV hierarchies. Since both in the latter and in the $N = 2$ superextensions a $U(1)$ current plays a crucial role, one may suspect that they might have much in common. This seems to be the case.

The paper is organized as follows. In section 2 we introduce our conjecture and explain how to make use of it. In section 3 we show that this conjecture holds true in the $N = 2$ superextensions of 2- and 3-KdV. In section 4 we introduce the supersymmetric Lax operator for the generalized NLS hierarchy and in section 5 we show that its bosonic limit reproduces one of the cases predicted by our conjecture. In section 6 we discuss the problem of the transformation between different pictures or bases, which are necessary in order to efficiently describe $N = 2$ superextensions.

2 The bosonic limit of $N = 2$ supersymmetric integrable hierarchies with $N = 2$ super W_n second hamiltonian structure

Let us first summarize the state of affairs concerning $N=2$ superextensions. Given a bosonic integrable hierarchy characterized by a certain number of fields, say an n -KdV hierarchy for definiteness, the corresponding $N=2$ superextension contains additional bosonic fields. Therefore, while the final aim is to construct for any such superextension the appropriate Lax operator, the first question one has to face is what bosonic fields (obeying specific flow equations) we need to add to have a system admitting $N = 2$ supersymmetrization¹. One of the possible ways to answer this question is to construct the $N = 2$ supersymmetric extension of the second hamiltonian structure of the bosonic system, as a first step in the construction of the entire system. Just in this way the $N = 2$ supersymmetric KdV equation [4], the $N = 2$ Boussinesq equation [5, 6], $N = 3, 4$ KdV equations [7, 8] and generalized KdV equation with $N = 2$ super W_4 algebra as second Hamiltonian structure [9], were found. Up to now these are the cases in which the $N=2$ superalgebra has been explicitly constructed, although an analogous construction for general W_n is only a matter of calculation.

¹Strictly speaking one may find many extensions: here we will consider only the minimal $N = 2$ superextensions, i.e. superextensions characterized by a minimal number of fields.

In this approach a certain amount of guesswork is necessary. Starting from the known second hamiltonian structure one writes down a general hamiltonian with free parameters and corresponding equations of motion. In all known cases integrability of the constructed $N = 2$ supersymmetric system holds only for some special values of these parameters. It is interesting that for the $N = 2$ super KdV [4], $N = 2$ super-Boussinesq [5, 6] and $N = 2$ extended 4-KdV equation [9] which possess $N = 2$ W_2 , W_3 and W_4 superalgebras as their second Hamiltonian structure, respectively, there are three values of the parameters giving rise to three integrable hierarchies in each case.

From the above it is clear that the bosonic sector of an $N=2$ superextension contains a good deal of information concerning the supersymmetric system. In this section we concentrate on the structure of bosonic sectors or limits (all fermionic fields are set to zero) of the $N = 2$ supersymmetric hierarchies with $N = 2$ super W_n algebras as their second Hamiltonian structure. For each n we construct three different bosonic systems and three different Lax operators, each of them leading to different integrable hierarchies. Then we formulate our *Conjecture* about the structure of the Lax operators for the bosonic sectors of the $N = 2$ supersymmetric integrable hierarchies with $N = 2$ super- W_n second hamiltonian structure and, finally, specify the method for fixing the arbitrary parameters in their $N = 2$ hamiltonians.

The $N = 2$ super W_{m+1} algebras can be obtained via hamiltonian reduction of the affine $sl(m+1|m)$ superalgebras with the set of constraints corresponding to the principal embedding of $sl(2|1)$ into $sl(m+1|m)$ [10]. By construction, the bosonic limit of the $N = 2$ W_{m+1} superalgebra is represented in a suitable basis by the direct sum algebra

$$W_{m+1} \oplus W_m \oplus U(1) \quad (2.1)$$

with the following relation among central charges

$$\frac{c_{W_{m+1}}}{c_{W_m}} = -\frac{m+2}{m-1}, \quad (m \geq 2). \quad (2.2)$$

Thus, after putting to zero all fermionic fields in the $N = 2$ super-($m+1$)-KdV equations (with $N = 2$ W_{m+1} superalgebra as its second hamiltonian structure), the bosonic equations will possess the algebra (2.1) as second hamiltonian structure². The question we address here is how many bosonic Lax operators providing the hamiltonian structure (2.1) can be constructed. The key to answer this question is provided by the analysis of the (n, m) -th KdV hierarchies of refs.[1, 2, 3].

Let us introduce the following n -KdV Lax operator

$$L_{[n;\alpha]} = (\alpha\partial)^n + \sum_{i=2}^n W_i^{(n)}(z) (\alpha\partial)^{n-i} \quad (2.3)$$

where α is a numerical parameter. This Lax operator induces, as second Hamiltonian structure on the currents $W_i^{(n)}(z)$, the W_n algebra with central charge

$$c_{W_n} = n(n^2 - 1)\alpha^2. \quad (2.4)$$

On the other hand, it was shown in [3] that the Lax operator $L_{[n,m;\alpha]}$ constructed from the $(n+m)$ -KdV and the m -KdV Lax operators (2.3) together with a $U(1)$ current $J(z)$, and

²The appearance of the graded structure of the $sl(m|n)$ algebra (its Cartan part) in the second hamiltonian structure of multi boson realization of KP hierarchy have been discussed in the papers [11, 12].

defined by

$$\begin{aligned}
L_{[n,m;\alpha]} &= \left(e^{\frac{m}{2}\partial^{-1}(J(z))} L_{[n+m;\alpha]} e^{-\frac{m}{2}\partial^{-1}(J(z))} \right) \left(e^{\frac{m+n}{2}\partial^{-1}(J(z))} L_{[m;\alpha]} e^{-\frac{m+n}{2}\partial^{-1}(J(z))} \right)^{-1} \\
&\equiv \left(\nabla_{(m)}^{n+m} + \sum_{i=2}^{n+m} W_i^{(n+m)}(z) \nabla_{(m)}^{n+m-i} \right) \left(\nabla_{(n+m)}^m - \sum_{i=2}^m W_i^{(m)}(z) \nabla_{(n+m)}^{m-i} \right)^{-1}, \quad (2.5)
\end{aligned}$$

where $\nabla_{(m)}$ is the covariant derivative

$$\nabla_{(m)} \equiv \alpha \left(\partial - \frac{m}{2} J(z) \right) \quad (2.6)$$

gives rise to the second hamiltonian structure

$$W_{n+m} \oplus W_m \oplus U(1), \quad \text{if } m \geq 2; \quad (2.7)$$

$$W_{n+1} \oplus U(1), \quad \text{if } m = 1; \quad (2.8)$$

$$W_n, \quad \text{if } m = 0. \quad (2.9)$$

These algebras have central charges

$$c_{W_{n+m}} = (n+m) \left[(n+m)^2 - 1 \right] \alpha^2, \quad c_{W_m} = -m(m^2 - 1) \alpha^2. \quad (2.10)$$

Let us notice, in particular, the identity

$$L_{[n,0;\alpha]} \equiv L_{[n;\alpha]}. \quad (2.11)$$

The flow equations and hamiltonians are defined in the standard way

$$\frac{\partial}{\partial t_r} L_{[n,m;\alpha]} = A \left[\left(L_{[n,m;\alpha]}^{\frac{r}{n}} \right)_+, L_{[n,m;\alpha]} \right], \quad (2.12)$$

$$\frac{\partial}{\partial t_r} W_i = \{ H_r^{(n,m)}, W_i \}, \quad (2.13)$$

$$H_r^{(n,m)} \equiv \int dx \operatorname{res} \left(L_{[n,m;\alpha]}^{\frac{r}{n}} \right), \quad (2.14)$$

where A is a normalization constant. The subscript $(+)$ means differential part of pseudo-differential operator and the usual definition of residue as the coefficient of ∂^{-1} is understood.

Now, using the Lax operators (2.5) and keeping in the mind (2.7)-(2.9) and the identity (2.11) we can construct three Lax operators, which are our candidates to reproduce the bosonic limit (2.1) of the $N = 2$ W_{m+1} superalgebra³

$$L_{[m;\alpha]}^{(1)} \equiv L_{[1,m;\alpha]} \Rightarrow W_{m+1} \oplus W_m \oplus U(1), \quad (2.15)$$

$$L_{[m;\alpha]}^{(2)} \equiv L_{[m,1;\alpha]} \oplus L_{[m,i;\alpha]} \Rightarrow (W_{m+1} \oplus U(1)) \oplus W_m, \quad (2.16)$$

$$L_{[m;\alpha]}^{(3)} \equiv L_{[m+1;\alpha]} \oplus L_{[m+1,i;\alpha]}^M \Rightarrow W_{m+1} \oplus (W_m \oplus U(1)), \quad (2.17)$$

³The forth possible Lax operator $L_{[m;\alpha]}^{(4)} \equiv L_{[m+1;\alpha]} \oplus L_{[m-1,1;i;\alpha]} \Rightarrow W_{m+1} \oplus (W_m \oplus U(1))$ with $m \geq 2$ produce systems which can not be recognize as bosonic limit of $N = 2$ hierarchies (at least for $m = 2$).

where ‘ i ’ is the imaginary unit and the sign \oplus between Lax operators in (2.16), (2.17) means that they act in different subspaces give rising to non-interacting systems. The Lax operator

$$L_{[n;\alpha]}^M \equiv \nabla_{(2(n-1))} \left(\nabla_{(-2)}^{n-1} + \sum_{i=2}^{n-1} W_i^{(n-1)}(z) \nabla_{(-2)}^{n-i-1} \right) \quad (2.18)$$

induces the W_n algebra realized in terms of W_{n-1} and $U(1)$ currents, see [13]. One can check that the relation (2.2) between the central charges of $N = 2$ W_{m+1} algebra is satisfied for the algebras (2.15)-(2.17) with the corresponding central charges (2.4), (2.10). Let us remark that for the special value $m = 1$ the Lax operator $L_{[1;\alpha]}^{(1)}$ is equivalent to $L_{[1;\alpha]}^{(2)}$.

Now we are ready to formulate our promised *Conjecture*: *the bosonic sectors of the $N = 2$ supersymmetric hierarchies with $N = 2$ W_{m+1} second hamiltonian structure can be described by the Lax operators (2.15)-(2.17); correspondingly we will have three different $N = 2$ hierarchies.*

We will substantiate this *Conjecture* later on with several examples and by constructing the super Lax operator corresponding to (2.15). For the time being let us remark that an interesting consequence of our *Conjecture* is the possibility to construct the superfield hamiltonians for the $N = 2$ supersymmetric hierarchies almost straightforwardly. Indeed, if we write the most general superfield expression for such hamiltonians in terms of the initial supercurrents, which form the $N = 2$ super W_{m+1} algebra, and put all fermionic fields equal to zero, then, after passing to the components in the basis (2.1), we can fix all the coefficients by simply comparing the resulting expression with the corresponding bosonic hamiltonian (2.14) following from the Lax operators (2.15)-(2.17).

Let us finally stress that among the Lax operators (2.15)-(2.17) only the first is ‘irreducible’, the remaining two being represented by the direct sum of two Lax operators. This means that, for the hierarchies induced by (2.16) and (2.17), the fields belonging to different irreducible addenda do not interact (they interact only via the fermionic fields in the $N = 2$ supersymmetric hierarchy). Thus in the bosonic limit, only the first bosonic system with Lax operator (2.15) is non trivial. This system is just the $(1, n)$ -KdV hierarchy[3]. We will construct the $N = 2$ supersymmetric Lax operator for the $N = 2$ supersymmetric extension of this system a bit later after showing, in the next section, that our *Conjecture* works in the cases of the $N = 2$ W_3 and $N = 2$ W_2 hierarchies.

3 Examples

In this section we will show how it is possible to reconstruct the $N = 2$ supersymmetric hamiltonians for $N = 2$ W_3 and W_2 hierarchies from their bosonic limits defined via our *Conjecture*.

3.1 The $N=2$ supersymmetric Boussinesq hierarchies

The $N = 2$ super Boussinesq equation for which the second hamiltonian structure is given by the classical $N = 2$ super- W_3 algebra [10], has been constructed in [14, 6]. It can be defined as the system of two $N = 2$ superfield equations for the supercurrents $J(Z), T(Z)$ having superspins equal to 1, 2, respectively

$$\frac{\partial T}{\partial t} = \{T, H_2\}, \quad \frac{\partial J}{\partial t} = \{J, H_2\} \quad (3.1)$$

with the hamiltonian H_2

$$H_2 = \int dZ \left(T + aJ^2 \right), \quad (3.2)$$

where $Z = (z, \theta, \bar{\theta})$ is coordinate of $N = 2$ superspace, $dZ = dzd\theta d\bar{\theta}$ and D, \bar{D} are the $N = 2$ supersymmetric fermionic covariant derivatives

$$D = \frac{\partial}{\partial \theta} - \frac{1}{2} \bar{\theta} \frac{\partial}{\partial z}, \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} - \frac{1}{2} \theta \frac{\partial}{\partial z}, \quad (3.3)$$

$$\{D, \bar{D}\} = -\frac{\partial}{\partial z}, \quad \{D, D\} = \{\bar{D}, \bar{D}\} = 0.$$

The Poisson brackets between J, T are defined to be $N = 2$ super- W_3 algebra. Explicitly, the $N = 2$ super Boussinesq equation reads as follows

$$\begin{aligned} \frac{\partial T}{\partial t} &= -2J''' - [D, \bar{D}] T' + \frac{80}{c} (\bar{D} J D J)' + \frac{32}{c} J' [D, \bar{D}] J + \frac{16}{c} J [D, \bar{D}] J' + \frac{256}{c^2} J^2 J' \\ &+ \left(\frac{40}{c} - 2a \right) \bar{D} J D T + \left(\frac{40}{c} - 2a \right) D J \bar{D} T + \left(\frac{64}{c} + 4a \right) J' T + \left(\frac{24}{c} + 2a \right) J T', \\ \frac{\partial J}{\partial t} &= 2T' - a \left(\frac{c}{4} [D, \bar{D}] J' - 4J J' \right), \end{aligned} \quad (3.4)$$

where $c = c_{W_3}$ is the central charge of the $N = 2$ super W_3 algebra. The integrability properties of the equation (3.4) have been studied in [5, 6] where it was shown that the first six conserved currents exist for the following three values of the parameter a

$$a^{(1)} = 20/c, \quad a^{(2)} = -16/c, \quad a^{(3)} = -4/c. \quad (3.5)$$

However the Lax operator has been found only for the value $a^{(3)}$.

Let us demonstrate how our *Conjecture* can be applied to the $N = 2$ super Boussinesq equation. From the second Hamiltonian structure of the bosonic sector and the Lax operators (2.15)-(2.17) we can immediately write down the expressions for the bosonic second order hamiltonian densities \mathcal{H}_2 (2.14)

$$\mathcal{H}_2^{(1)} = u_2 - \frac{48}{c} u_1 J_B - \frac{720}{c^2} J_B^3 - \frac{72}{c} v_1 J_B, \quad (3.6)$$

$$\mathcal{H}_2^{(2)} = u_2 + \frac{24}{c} u_1 J_B - \frac{576}{c^2} J_B^3, \quad (3.7)$$

$$\mathcal{H}_2^{(3)} = u_2 - \frac{24}{c} v_1 J_B - \frac{144}{c^2} J_B^3, \quad (3.8)$$

where the currents u_2, u_1 form the W_3 algebra

$$\begin{aligned} \{u_1(z_1), u_1(z_2)\} &= \left(-\frac{c}{12} \partial^3 + u_1 \partial + \partial u_1 \right) \delta(z_1 - z_2), \\ \{u_1(z_1), u_2(z_2)\} &= \left(\frac{c}{24} \partial^4 + u_2 \partial + 2\partial u_2 - \partial^2 u_1 \right) \delta(z_1 - z_2), \\ \{u_2(z_1), u_2(z_2)\} &= \left(\frac{c}{36} \partial^5 + \partial^2 u_2 - u_2 \partial^2 + \frac{16}{c} u_1 \partial u_1 - \frac{2}{3} u_1 \partial^3 - \frac{2}{3} \partial^3 u_1 \right) \delta(z_1 - z_2), \end{aligned} \quad (3.9)$$

while v_1 and J_B span $W_2 \oplus U(1)$

$$\begin{aligned} \{v_1(z_1), v_1(z_2)\} &= \left(\frac{c}{48} \partial^3 + v_1 \partial + \partial v_1 \right) \delta(z_1 - z_2), \\ \{J_B(z_1), J_B(z_2)\} &= -\frac{c}{36} \delta'(z_1 - z_2), \end{aligned} \quad (3.10)$$

where derivatives and currents appearing in the right hand side are understood to be evaluated at z_1 . We have fixed the normalization of the hamiltonians by setting equal to 1 the coefficient of the spin 3 current u_2 . Let us note that it is straightforward to pass from the currents in terms of which the Lax operators (2.15)-(2.17) are defined, to the currents u_2, u_1, v_1, J_B . For example for the Lax operator $L_{[2;\alpha]}^{(2)}$ (2.16), which can be represented in the following equivalent form

$$L_{[2;\alpha]}^{(2)} = \left(\alpha^2 \partial^2 + w_1 + w_2 \frac{1}{\alpha \partial - S_1} \right) \oplus \left(-\alpha^2 \partial^2 + w_0 \right), \quad (3.11)$$

these transformations (up to an automorphism $J_B \Rightarrow -J_B$, $u_2 \Rightarrow -u_2 + u'_1$, $u_1 \Rightarrow u_1$ of the algebras (3.9), (3.10)) are

$$\begin{aligned} w_0 &= v_1, & S_1 &= -\frac{3}{2} J_B, \\ w_1 &= u_1 - \frac{18}{c} J_B^2 - \frac{3}{2} J'_B, \\ w_2 &= u_2 + \frac{24}{c} u_1 J_B - \frac{576}{c^2} J_B^3 - \frac{72}{c} J_B J'_B - J''_B. \end{aligned} \quad (3.12)$$

To compare the bosonic limit of the $N = 2$ superfield hamiltonian (3.2) with (3.6)-(3.8), we must first integrate over $\theta, \bar{\theta}$ and put all fermionic fields equal to zero

$$H_2 \Rightarrow H_2^B = \int dz \left([D, \bar{D}] T | + 2aJ [D, \bar{D}] J | \right), \quad (3.13)$$

where $|$ denotes the $\theta, \bar{\theta}$ independent part (i.e. the limit $\theta = \bar{\theta} = 0$), and then pass to the bosonic components u_1, u_2, v_1, J_B which form the $W_3 \oplus W_2 \oplus U(1)$ (3.9), (3.10). In the case at hand the correspondence is (up to the above mentioned automorphism of the algebras (3.9), (3.10))

$$\begin{aligned} J | &= 3J_B, \\ [D, \bar{D}] J | &= 2u_1 + 2v_1 - \frac{36}{c} J_B^2, \\ T | &= -u_1 - 4v_1, \\ [D, \bar{D}] T | &= -6u_2 + 3u'_1 + \frac{48}{c} (u_1 + 4v_1) J_B. \end{aligned} \quad (3.14)$$

Using this we get the following expression for H_2^B

$$H_2^B = -6 \int dz \left(u_2 - \frac{8}{c} \left(1 + \frac{ac}{4} \right) u_1 J_B - \frac{16}{c} \left(2 + \frac{ac}{8} \right) v_1 J_B + \frac{36a}{c} J_B^3 \right). \quad (3.15)$$

It is easy to check that the \mathcal{H}_2^B (3.15) coincides (up to an inessential rescaling) with $\mathcal{H}_2^{(1)}, \mathcal{H}_2^{(2)}, \mathcal{H}_2^{(3)}$ (3.6)-(3.8), exactly for the values of parameter a (3.5), respectively.

We stress that our *Conjecture* and the procedure we have proposed do not guaranteed the integrability and the conservation of the hamiltonians we have constructed. It provides a necessary but not sufficient condition for integrability. The safest and most timesaving way to prove integrability is through explicit construction of Lax operators. Unfortunately this is very complicated even for the $N = 2$ W_3 hierarchies. The main problem is that the variables (supercurrents) in which the second hamiltonian structure is transparent are not very suitable

for writing the Lax operators. As an example we present here the possible Lax operators for the $N = 2$ super Boussinesq which correspond to the following values of parameters: $a^{(1)} = 20/c$, $a^{(2)} = -16/c$.

The starting point is the following general form of the Lax operators

$$L = \partial + \kappa_0 + \sum_{i=1}^{\infty} \left(\kappa_i + \sigma_i \bar{D} + \bar{\sigma}_i D + \rho_i [D, \bar{D}] \right) \partial^{-i}, \quad (3.16)$$

where $\kappa, \sigma, \bar{\sigma}$ and ρ are sums of monomials constructed from supercurrents $J(Z), T(Z)$ with the proper dimensions. We will find out that the flow equations can appear in the two different forms

$$\frac{\partial}{\partial t_n} L = A \left[(L^n)_+, L \right] \quad (3.17)$$

or

$$\frac{\partial}{\partial t_n} L = B \left[(L^n)_{\geq 1}, L \right], \quad (3.18)$$

where the subscript $(+)$ means differential part of a pseudodifferential operator, while (≥ 1) means the pure differential part with exclusion of the constant term. We do not know from the start which flow equations we have to use in order to reproduce the equations (3.4), so we have to check all the possibilities. The results of our calculations can be summarized as follows

$$\underline{a^{(1)} = 20/c.}$$

$$\begin{aligned} A = -1, L_{(1)} &= \frac{c}{8} \partial + 27 [\bar{D} D J] \partial^{-1} - 27 \left[\bar{D} D \left(U + \frac{36}{c} J^2 - \frac{5}{2} J' \right) \right] \partial^{-2} \\ &+ \left[\bar{D} D \left(-54 U' + 108 J'' + \frac{1944}{c} J U + \frac{46656}{c^2} J^3 \right. \right. \\ &\left. \left. - \frac{972}{c} J J' + \frac{5832}{c} J \bar{D} D J \right) - \frac{5832}{c} [\bar{D} D J] \bar{D} D J \right] \partial^{-3} + \dots, \end{aligned} \quad (3.19)$$

$$A = -3, L_{(2)} = \frac{c}{8} \partial - 3 [D \bar{D} J] \partial^{-1} - \left[D \bar{D} \left(U + \frac{36}{c} J^2 - \frac{3}{2} J' \right) \right] \partial^{-2} + \dots, \quad (3.20)$$

where $U = T + \frac{1}{2} [D, \bar{D}] J - \frac{16}{c} J^2$ and the flow equations are the (3.17). (The square brackets mean that the derivatives act only on the terms inside brackets.)

$$\underline{a^{(2)} = -16/c.}$$

$$\begin{aligned} B = -3, L_{(1)} &= \frac{c}{8} \partial + \frac{3}{2} J + \frac{3}{2} \bar{D} \partial^{-1} [D J] - \frac{1}{2} \partial^{-1} \tilde{U} \\ &+ \left(\frac{1}{2} [D \tilde{U}] \bar{D} + \frac{6}{c} J \tilde{U} + \frac{1}{4} [[D, \bar{D}] \tilde{U}] - \frac{9}{c} [D J] [\bar{D} J] \right) \partial^{-2} + \dots, \end{aligned} \quad (3.21)$$

$$B = -3, L_{(2)} = \frac{c}{8} \partial - \frac{3}{2} D J \bar{D} \partial^{-1} + \frac{1}{2} D \left(\tilde{U} + \frac{3}{2} [[D, \bar{D}] J] + \frac{3}{2} J' \right) \bar{D} \partial^{-2} + \dots, \quad (3.22)$$

where $\tilde{U} = T + \frac{1}{2} [D, \bar{D}] J + \frac{2}{c} J^2$ and the Lax equation is given by (3.18).

Above we have reported only the first few terms of the Lax operators, but we have checked them as far as the ∂^{-5} terms. We did not succeed in finding a closed form for these Lax

operators. On the other hand in the next section we will present the Lax operator for the case $a^{(1)} = 20/c$, using new variables which renders its structure more transparent.

In the rest of this section we discuss the problem of extracting the hamiltonians from the Lax operators (3.19)-(3.22).

First of all, let us note that the Lax operator (3.22) contains the standard residue, i.e. the coefficient before $[D, \overline{D}]\partial^{-1}$ [4]. Therefore in this case the hamiltonians can be written down as

$$H_n = \int dZ \operatorname{res}(L^n). \quad (3.23)$$

The Lax operator (3.21) does not have the standard residue (which vanishes). We can in such a case apply the definition of residues as the constant part of the pseudo-differential operator, see [15]. We checked that the first four hamiltonians can be constructed following this definition, i.e.

$$H_n = \int dZ (L^n)_0, \quad (3.24)$$

As for the Lax operators (3.19), (3.20), they do not contain neither the latter nor the standard $N = 2$ residues [4]. Moreover, these Lax operators look like pure bosonic ones, since they do not contain spinor D or \overline{D} operators acting separately. It is very interesting that the corresponding hamiltonians (at least the first four ones) can be obtained in these cases using the definition of residues for bosonic pseudodifferential operators, i.e. as the integrated coefficients of ∂^{-1}

$$H_n = \int dz \operatorname{res}_B(L^n), \quad (3.25)$$

where the integration is over the space coordinate z . The main reason why this definition works is that the coefficients before ∂^{-1} in L^n (3.19)-(3.20), at least as far as $n = 4$, can be represented as

$$\operatorname{res}_B(L^n) = [D, \overline{D}] (\mathcal{H}_n) + \text{full space derivative terms}. \quad (3.26)$$

The proof that our definitions of hamiltonians, or equivalently of the so-called residues, work in the case of general degenerated $N = 2$ pseudodifferential operators, for which the standard residues are equal to zero, is an open problem which goes beyond the scope of this paper. However the above explicit calculations seem to indicate that proper alternative residues can always be defined.

3.2 The N=2 supersymmetric KdV hierarchies

The $N = 2$ super KdV equation

$$j = -j''' + \frac{36}{c} (J [D, \overline{D}] J)' - \frac{ac + 12}{2c} ([D, \overline{D}] J^2)' + \frac{36a}{c} J^2 J' \quad (3.27)$$

has been constructed in [4] starting from the Hamiltonian

$$H_3 = \int dZ \left(J [D, \overline{D}] J + \frac{a}{3} J^3 \right) \quad (3.28)$$

and the Poisson brackets between the supercurrent $J(Z)$ are supposed to be $N = 2$ superconformal algebra [16]

$$\{J(Z_1), J(Z_2)\} = \left(-\frac{c}{24} [D, \overline{D}] \partial + \overline{D} J D + D J \overline{D} + J \partial + \partial J \right) \delta(Z_1 - Z_2), \quad (3.29)$$

where

$$\delta(Z_1 - Z_2) = (\theta_1 - \theta_2)(\bar{\theta}_1 - \bar{\theta}_2)\delta(z_1 - z_2)$$

is the delta function in $N = 2$ superspace, and derivatives and supercurrents appearing in the right hand side are understood to be evaluated at Z_2 . This equation is proved to be integrable for the following values of parameter a [4, 17]

$$a^{(1)} = a^{(2)} = -48/c, \quad a^{(3)} = 24/c, \quad a^{(0)} = -12/c. \quad (3.30)$$

Then the three bosonic Lax operators (2.15)-(2.17) reduce to two independent ones (see discussion below formula (2.18)). Following the same arguments as for $N = 2$ super W_3 , we can find the third hamiltonians for these bosonic hierarchies in terms of the currents v_1, J_B forming the $W_2 \oplus U(1)$ algebra (3.10)

$$\mathcal{H}_3^{(1)} = \mathcal{H}_3^{(2)} = v_1^2 - \frac{108}{c}v_1J_B^2 - \frac{3}{4}J_BJ_B'' + \frac{1620}{c^2}J_B^4, \quad (3.31)$$

$$\mathcal{H}_3^{(3)} = v_1^2 - \frac{3}{4}J_BJ_B'' - \frac{324}{c^2}J_B^4. \quad (3.32)$$

Now, as in the case of the $N = 2$ super Boussinesq equation, we integrate over $\theta, \bar{\theta}$ in (3.28) and pass to the components v_1, J_B

$$v_1 = -\frac{1}{2} [D, \bar{D}] J| - \frac{6}{c} J^2|, \quad J_B = \frac{i}{\sqrt{3}} J| \quad (3.33)$$

with the Poisson brackets (3.10). It is easy to check that the bosonic limit of (3.28), written in terms of v_1, J_B (3.33),

$$\mathcal{H}_3^B = 4 \left(v_1^2 + \frac{3}{2} \left(a - \frac{24}{c} \right) v_1 J_B^2 - \frac{3}{4} J_B J_B'' - \frac{27}{c} \left(a - \frac{12}{c} \right) J_B^4 \right) \quad (3.34)$$

coincides with (3.31), (3.32) (up to an inessential rescaling) for $a^{(1)} = a^{(2)} = -48/c$ and $a^{(3)} = 24/c$, respectively. The case $a^{(0)} = -12/c$ is exceptional and cannot be described within the scheme of our *Conjecture* (this case has been discussed in [17]).

To close this section let us stress that the described procedure is certainly applicable not only to the first meaningful hamiltonians but also to the few subsequent ones. We can retrieve the hamiltonians from the bosonic parts for which the Lax operators (and thus the hamiltonians) are defined by (2.15)-(2.17). Based on our *Conjecture* we can also claim that for the $N = 2$ hierarchies with bosonic limit given by (2.17) there are no supersymmetric hamiltonians of order $k(m+1), k = 1, 2, \dots$ because they do not exist in the bosonic case ($H_n = \int dz \text{res} ((L_{[m+1, \alpha]}^{(3)})^{n/(m+1)}) = 0$ for $n = k(m+1)$ since $L_{[m+1, \alpha]}^{(3)}$ is a pure differential operator). We have checked this for all known six hamiltonians in $N = 2$ W_2, W_3 and W_4 cases. However our procedure is expected to fail if the hamiltonian under investigation contains superfield terms which after integration over $\theta, \bar{\theta}$ disappear in the bosonic limit. The coefficients of such terms cannot be fixed in the framework of our *Conjecture*. The first appearance of such terms might take place in H_8 ($\int dZ D J \bar{D} J D J' \bar{D} J'$ term) but it is unclear now whether such terms do occur in hamiltonians. It might be that the coefficients of such terms vanish in hamiltonians of $N = 2$ supersymmetric integrable hierarchies. If this is true, one can reconstruct all hamiltonians from their bosonic limit.

4 The $N=2$ supersymmetric GNLS hierarchies

In this section we construct the $N = 2$ supersymmetric generalization of the bosonic generalized non-linear Schrödinger (GNLS) hierarchies and their manifestly $N = 2$ supersymmetric Lax operators. In the following section we will recognize that their bosonic limits coincide with the hierarchies defined by the Lax operators (2.15) (in fact a direct sum of two of them).

Let us introduce $(n+m)$ pairs of chiral and anti-chiral $N = 2$ superfields $F_A(Z)$ and $\bar{F}_A(Z)$ with capital Latin indices $A, B = 1, \dots, n+m$

$$DF_A(Z) = \bar{D}\bar{F}_A(Z) = 0, \quad (4.1)$$

which are fermionic for $A = 1, \dots, n$ and bosonic for $A = n+1, \dots, n+m$. The grading is $F_A F_B = (-1)^{d_A d_B} F_B F_A$, where $d_A = 1$ ($d_A = 0$) for fermionic (bosonic) superfields. In what follows, we find it convenient to denote the indices of the fermionic superfields by small Latin letters $i, j = 1, \dots, n$ and the bosonic ones by Greek letters $\alpha, \beta = 1, \dots, m$, respectively⁴.

Our ansatz for the Lax operator of the $N = 2$ super GNLS hierarchy is

$$L = \partial - \frac{1}{2} F_A(Z) \bar{F}_A(Z) - \frac{1}{2} F_A(Z) \bar{D} \partial^{-1} [D \bar{F}_A(Z)], \quad (4.2)$$

where summation over repeated indices is understood and the square brackets mean that the fermionic derivative D act only on the term \bar{F}_A inside brackets. Such operator provides (for integer k) the consistent flows

$$\frac{\partial}{\partial t_k} L = [(L^k)_{\geq 1}, L], \quad (4.3)$$

where the subscript ≥ 1 means that only the purely derivative part must be considered. Let us note that only the spin of the product $F_A \bar{F}_A$ is fixed by the ansatz (4.2), and it is equal to 1.

The first flow from (4.3) is trivial

$$\begin{aligned} \frac{\partial}{\partial t_1} F_A &= F'_A, \\ \frac{\partial}{\partial t_1} \bar{F}_A &= \bar{F}'_A, \end{aligned} \quad (4.4)$$

while the second reads

$$\begin{aligned} \frac{\partial}{\partial t_2} F_A &= F''_A + D(F_B \bar{F}_B \bar{D} F_A), \\ \frac{\partial}{\partial t_2} \bar{F}_A &= -\bar{F}''_A + \bar{D}(F_B \bar{F}_B D \bar{F}_A). \end{aligned} \quad (4.5)$$

Beside global $N = 2$ supersymmetry the Lax operator (4.2) and the flows (4.3) are invariant with respect to the $GL(n|m)$ supergroup. Let us note that the second flow (4.5) in the case $n = 1, m = 0$, which corresponds to one pair of fermionic chiral-anti-chiral superfields, is just the $N = 2$ super NLS equation [18]. The Lax operator (4.2) generalizes to the multi-components case (i.e., for arbitrary values of n and m) the Lax operator for $N = 2$ super NLS hierarchy [15]. Moreover, as we will show in the next section, the bosonic limit of these systems coincides with bosonic GNLS hierarchies. That is why we call the corresponding hierarchies the $N = 2$ super GNLS hierarchies.

⁴For simplicity we call bosonic (fermionic) the commuting (anticommuting) fields. However they may not have the usual spin-statistics connection. In the latter case we can perhaps identify them with BRST ghosts.

Let us write down the explicit expressions for the first three hamiltonians

$$\begin{aligned}
H_1 &= -\frac{1}{2} \int dZ \left(F_A \bar{F}_A \right) , \\
H_2 &= \int dZ \left(F_A \bar{F}'_A + \frac{1}{4} (F_A \bar{F}_A)^2 \right) , \\
H_3 &= -\frac{3}{2} \int dZ \left(F_A \bar{F}''_A - \frac{1}{2} \bar{D} (F_A \bar{F}_A) \cdot D (F_B \bar{F}_B) + F_A \bar{F}'_A \cdot F_B \bar{F}_B + \frac{1}{12} (F_A \bar{F}_A)^3 \right) .
\end{aligned} \tag{4.6}$$

It is interesting to remark that the first hamiltonian density \mathcal{H}_1 satisfy the following equation of motion

$$\frac{\partial}{\partial t_2} \mathcal{H}_1 = (\mathcal{H}'_1 + \mathcal{H}_2)' , \tag{4.7}$$

so one can find the additional integral of motion

$$\tilde{H}_1 = -\frac{1}{2} \int dz \left(F_A \bar{F}_A \right) , \tag{4.8}$$

where we have only space integration.

Note that the standard definition of residue in the case of $N = 2$ supersymmetric pseudodifferential operators as the coefficient of $[D, \bar{D}] \partial^{-1}$, once again cannot be applied to our Lax operator (4.2), due to vanishing of such residues for any power of L . Nevertheless, the infinite number of conserved currents can be extracted from L (4.2) as follows

$$H_k = \int dZ (L^k)_0 , \tag{4.9}$$

where the subscript 0 means the constant part of the operator. We do not have a general proof that for an arbitrary degenerate $N = 2$ supersymmetric pseudodifferential operator the constant part gives the correct hamiltonians, but for the case at hand this statement has been explicitly checked for the hamiltonians (4.6).

It is instructive to rewrite the second flow equations (4.5) in terms of unconstrained $N = 1$ superfields $\mathcal{F}_A(\tilde{Z}), \bar{\mathcal{F}}_A(\tilde{Z})$, where $\tilde{Z} = (z, \theta_2)$ is the coordinate of the $N = 1$ real superspace. This can be done by solving the chirality conditions (4.1) via superfields $\mathcal{F}_A(\tilde{Z}), \bar{\mathcal{F}}_A(\tilde{Z})$

$$\begin{aligned}
F_A(Z) &= \mathcal{F}_A(\tilde{Z}) + i\theta_1 D_2 \mathcal{F}_A(\tilde{Z}), \\
\bar{F}_A(Z) &= \bar{\mathcal{F}}_A(\tilde{Z}) - i\theta_1 D_2 \bar{\mathcal{F}}_A(\tilde{Z}) .
\end{aligned} \tag{4.10}$$

Here $\theta_1 \equiv \frac{\theta + \bar{\theta}}{2}, \theta_2 \equiv \frac{\theta - \bar{\theta}}{2i}$ are real Grassmann coordinates and $N = 1$ supersymmetric fermionic covariant derivatives D_1, D_2 are defined by

$$D_1 \equiv D + \bar{D} = \frac{\partial}{\partial \theta_1} - \theta_1 \frac{\partial}{\partial z} , \quad D_2 \equiv i(D - \bar{D}) = \frac{\partial}{\partial \theta_2} - \theta_2 \frac{\partial}{\partial z} , \tag{4.11}$$

$$D_1^2 = D_2^2 = -\frac{\partial}{\partial z} , \quad \{D_1, D_2\} = 0.$$

Substituting (4.10) into (4.5) and using (4.11) we get the following second flow equations

$$\begin{aligned}
\frac{\partial}{\partial t_2} \mathcal{F}_A &= \mathcal{F}''_A + (-1)^{d_B} \mathcal{F}_B D_2 (\bar{\mathcal{F}}_B D_2 \mathcal{F}_A), \\
\frac{\partial}{\partial t_2} \bar{\mathcal{F}}_A &= -\bar{\mathcal{F}}''_A + \bar{\mathcal{F}}_B D_2 (\mathcal{F}_B D_2 \bar{\mathcal{F}}_A) .
\end{aligned} \tag{4.12}$$

The systems (4.12) still possess $N = 2$ supersymmetry. However the latter is hidden due to the description by means of $N = 1$ superfields. The simplest case with $n = 1, m = 0$ has been considered in [19, 20] and called there the $N = 1$ NLS equation. Now, generalizing the Lax operator for $N = 1$ NLS [20], we can construct the Lax operator in terms of $N = 1$ superfields for the system (4.12)

$$L = \partial - \frac{1}{2}\mathcal{F}_A(\tilde{Z})\overline{\mathcal{F}}_A(\tilde{Z}) - \frac{1}{2}\mathcal{F}_A(\tilde{Z})D_2\partial^{-1}\left[D_2\overline{\mathcal{F}}_A(\tilde{Z})\right]. \quad (4.13)$$

This representation in terms of $N = 1$ superfields might be more convenient to analyze hamiltonian reduction from affine $N = 1$ superalgebras.

5 Bosonic limit of the N=2 supersymmetric GNLS hierarchies

In the $N = 2$ supersymmetric extended KP hierarchies we may select two main different bases (i.e., choices of elementary fields). It was first recognized in [18, 15] that while in one basis the second Hamiltonian structure of $N = 2$ supersymmetric extensions is transparent, in the second one the Lax operator can be written in an elegant and economic way. In the previous section we constructed the $N = 2$ supersymmetric extension of the GNLS hierarchies in the basis where their Lax operator structure is clear. However it is not evident to which bosonic systems we considered in section 2 these equations correspond. In this section we consider the bosonic limit of $N = 2$ hierarchies with the Lax operators (4.2) and establish their relations with the $(1, n)$ -KdV hierarchies.

To consider the bosonic limit of the second flow equations (4.5), let us define the components of the fermionic superfields as

$$f_i = \frac{1}{\sqrt{2}}\overline{D}F_i|, \quad \overline{f}_i = \frac{1}{\sqrt{2}}D\overline{F}_i|, \quad \psi_i = \frac{1}{\sqrt{2}}F_i|, \quad \overline{\psi}_i = \frac{1}{\sqrt{2}}\overline{F}_i| \quad (5.1)$$

and the components of the bosonic superfields as

$$\xi_\alpha = \frac{1}{\sqrt{2}}\overline{D}F_\alpha|, \quad \overline{\xi}_\alpha = \frac{1}{\sqrt{2}}D\overline{F}_\alpha|, \quad b_\alpha = \frac{1}{\sqrt{2}}F_\alpha|, \quad \overline{b}_\alpha = \frac{1}{\sqrt{2}}\overline{F}_\alpha|, \quad (5.2)$$

where $|$ means the $(\theta, \overline{\theta}) \rightarrow 0$ limit. So, $\psi_i, \overline{\psi}_i, \xi_\alpha, \overline{\xi}_\alpha$ are fermionic fields while $f_i, \overline{f}_i, b_\alpha, \overline{b}_\alpha$ are bosonic ones.

In terms of such components the equations (4.5) become

$$\begin{aligned} \frac{\partial}{\partial t_2} \begin{pmatrix} \psi_i \\ b_\alpha \end{pmatrix} &= \begin{pmatrix} \psi_i \\ b_\alpha \end{pmatrix}'' + 2(-\psi_j \overline{f}_j + b_\beta \overline{\xi}_\beta) \begin{pmatrix} f_i \\ b_\alpha \end{pmatrix} - 2(\psi_j \overline{\psi}_j + b_\beta \overline{b}_\beta) \begin{pmatrix} \psi_i \\ b_\alpha \end{pmatrix}', \\ \frac{\partial}{\partial t_2} \begin{pmatrix} f_i \\ \xi_\alpha \end{pmatrix} &= \begin{pmatrix} f_i \\ \xi_\alpha \end{pmatrix}'' + 2(-f_j \overline{f}_j - \psi_j \overline{\psi}_j' - b_\beta \overline{b}_\beta' + \xi_\beta \overline{\xi}_\beta) \begin{pmatrix} f_i \\ \xi_\alpha \end{pmatrix} \\ &\quad - 2(\psi_j \overline{\psi}_j + b_\beta \overline{b}_\beta) \begin{pmatrix} f_i \\ \xi_\alpha \end{pmatrix}' + 2(-\psi_j \overline{f}_j - \overline{\psi}_j f_j + b_\beta \overline{\xi}_\beta - \overline{b}_\beta \xi_\beta) \begin{pmatrix} \psi_i \\ b_\alpha \end{pmatrix}'. \end{aligned} \quad (5.3)$$

The equations for the components $\overline{\psi}_i, \overline{f}_i, \overline{b}_\alpha, \overline{\xi}_\alpha$ can be obtained from (5.3) by conjugation (t_2 is pure imaginary and under complex conjugation $t_2^* = -t_2$).

To get the bosonic limit we have to put all the fermionic fields $\psi_i, \bar{\psi}_i, \xi_\alpha, \bar{\xi}_\alpha$ to zero. This leaves us with the following set of bosonic equations

$$\begin{aligned}\frac{\partial}{\partial t_2} f_i &= f_i'' - 2f_i f_j \bar{f}_j - 2b_\beta (\bar{b}_\beta f_i)', \\ \frac{\partial}{\partial t_2} \bar{f}_i &= -\bar{f}_i'' + 2\bar{f}_i f_j \bar{f}_j - 2\bar{b}_\beta (b_\beta \bar{f}_i)',\end{aligned}\tag{5.4}$$

$$\begin{aligned}\frac{\partial}{\partial t_2} b_\alpha &= b_\alpha'' - 2b_\beta \bar{b}_\beta b_\alpha', \\ \frac{\partial}{\partial t_2} \bar{b}_\alpha &= -\bar{b}_\alpha'' - 2b_\beta \bar{b}_\beta \bar{b}_\alpha'.\end{aligned}\tag{5.5}$$

Let us note that in (5.4), (5.5) only the spins of the products $f_j \bar{f}_j$ and $b_\beta \bar{b}_\beta$ are fixed and they are equal to 2 and 1, respectively.

After passing to the new fields g_i, \bar{g}_i defined by

$$\begin{aligned}g_i &= f_i \exp(-\partial^{-1}(b_\beta \bar{b}_\beta)), \\ \bar{g}_i &= \bar{f}_i \exp(\partial^{-1}(b_\beta \bar{b}_\beta))\end{aligned}\tag{5.6}$$

we can rewrite the equations (5.4) as

$$\begin{aligned}\frac{\partial}{\partial t_2} g_i &= g_i'' - 2g_i g_j \bar{g}_j, \\ \frac{\partial}{\partial t_2} \bar{g}_i &= -\bar{g}_i'' + 2\bar{g}_i g_j \bar{g}_j.\end{aligned}\tag{5.7}$$

The fields b_α, \bar{b}_α are completely decoupled and the set of equations (5.7) form the GNLS equations [21]. They can be viewed as the second flow of GNLS hierarchies with the Lax operators \tilde{L}_1

$$\tilde{L}_1 = \partial - g_i \partial^{-1} \bar{g}_i\tag{5.8}$$

and the flow equations

$$\frac{\partial}{\partial t_k} L = [(L^k)_+, L].\tag{5.9}$$

As for the b_α, \bar{b}_α fields, which obey the closed set of equations (5.5), after passing to the new fields r_α, \bar{r}_α

$$\begin{aligned}r_\alpha &= b'_\alpha \exp(-\partial^{-1}(b_\beta \bar{b}_\beta)), \\ \bar{r}_\alpha &= \bar{b}_\alpha \exp(\partial^{-1}(b_\beta \bar{b}_\beta)),\end{aligned}\tag{5.10}$$

we get the following system

$$\begin{aligned}\frac{\partial}{\partial t_2} r_\alpha &= r_\alpha'' - 2r_\alpha r_\beta \bar{r}_\beta, \\ \frac{\partial}{\partial t_2} \bar{r}_\alpha &= -\bar{r}_\alpha'' + 2\bar{r}_\alpha r_\beta \bar{r}_\beta,\end{aligned}\tag{5.11}$$

which coincides with the GNLS equations. We remark that the transformations (5.10) look like formulas for ‘bosonization of bosons’.

Thus the set of equations (5.4), (5.5), which represent the bosonic limit of the $N = 2$ supersymmetric equations (4.5), can be converted to the form (5.7), (5.11) which is nothing but the direct sum of two GNLS systems [21]. But for this system, the transformations to

the $(1, n)$ -KdV hierarchies we considered in section 2 are well known [22, 3]. Thus we can conclude that the $N = 2$ supersymmetric hierarchies with Lax operators (4.2) are the $N = 2$ supersymmetric extensions of the direct sum of $(1, n)$ -KdV and $(1, m)$ -KdV hierarchies with the Lax operator $L_{[n;\alpha]}^{(1)} \oplus L_{[m;\alpha]}^{(1)}$ (2.15). For the special cases with only bosonic ($n = 0$) or fermionic ($m = 0$) superfields in the Lax operator (4.2) we have supersymmetric extensions of $(1, m)$ - or $(1, n)$ -KdV hierarchies, respectively. Thus there are two possibilities to construct $N = 2$ supersymmetric extension of $(1, n)$ -KdV hierarchies, based on bosonic or fermionic superfields. One expects that these superextensions be somehow related. As an example, in the next section we will estimate the explicit relation between $N = 2$ supersymmetric extensions of the $(1, 1)$ -KdV hierarchy with fermionic and bosonic superfields.

Let us point out that, despite the existence of the transformations (5.10), that bring the systems of equations for b_α, \bar{b}_α (5.5) to the GLNS system (5.11), the (5.5) system seems to be more fundamental. In fact the system (5.5) possesses the conserved current

$$\mathcal{H}_0 = b_\alpha \bar{b}_\alpha \quad (5.12)$$

which is missing for the (5.11) system. For completeness we present here the Lax operator for the b_α, \bar{b}_α system

$$\tilde{L}_2 = \partial - b'_\alpha (\partial + b_\beta \bar{b}_\beta)^{-1} \bar{b}_\alpha. \quad (5.13)$$

Finally we would like to stress that an alternative approach to the construction of $N = 2$ supersymmetric hierarchies recently proposed in [23] seems to be less general than ours. The author uses general superfields (instead of chiral ones, (4.1)) and a completely different ansatz for the super Lax operators (instead of ours, (4.2)). The absence of a chirality condition leads to a $4(n+m)$ component system in the bosonic limit (instead of a $2(n+m)$ one, as in our case). As a consequence the systems constructed in this and the previous sections for odd values of $n+m = 2l-1, l = 1, 2, \dots$, cannot be reproduced within the approach of [23]. For the other cases, one should prove that the bosonic limits do reproduce GLNS hierarchies, also beyond the case $n = 1, m = 0$, which is the only one proved in [23]. If this is true, it would be interesting to find the relation with our $N = 2$ supersymmetric hierarchies (for even values of $n+m$).

6 Transformations between bases

In the previous sections we showed that there are two preferred bases in which $N = 2$ supersymmetric extensions of $(1, n)$ -KdV hierarchies can be constructed. In one basis the second hamiltonian structure is transparent and it coincides with $N = 2$ super W_n algebra, while the structure of Lax operators is complicated. In the second basis the Lax operators can be written in the explicit form (4.2), but the second hamiltonian structure is hidden. For the bosonic limit, when all fermionic fields equal zero, the transformations between these bases are known [22]. But in the case of $N = 2$ supersymmetric extensions these transformations are more complicated. In this section we present some preliminary results concerning these transformations for the $N = 2$ super-KdV and super-Boussinesq hierarchies.

6.1 $N=2$ super-KdV case ($a^{(1)} = -48/c$)

The $N = 2$ super-KdV equation ($N = 2$ $(1, 1)$ -KdV) has the $N = 2$ superconformal algebra (3.29) as the second hamiltonian structure [4], and corresponds to one pair of bosonic

$B(Z), \bar{B}(Z)$ ($n = 0, m = 1$) or fermionic $F(Z), \bar{F}(Z)$ ($n = 1, m = 0$) chiral-anti-chiral superfields in the second basis with Lax operator given by (4.2). The transformation from $F(Z), \bar{F}(Z)$ to the single spin 1 $N = 2$ superfield $J(Z)$, which obeys the standard $N = 2$ super-KdV hierarchy equations, has been found in [18]

$$J = \frac{c}{24} \left(\frac{1}{2} F \bar{F} + \frac{\partial}{\partial z} (\ln D \bar{F}) \right). \quad (6.1)$$

It is interesting that the second hamiltonian structure for the F, \bar{F} fields coincides with $N = 2$ super $\hat{U}(2)$ algebra and the hamiltonians belong to the coset [15]

$$\frac{N = 2 \text{ super } \hat{U}(2)}{U(1) \oplus U(1)}.$$

In the second case, when we deal with bosonic superfields $B(Z), \bar{B}(Z)$, the transformation to the superfield $J(Z)$ has the following form

$$J = -\frac{c}{24} \left(\frac{1}{2} B \bar{B} + \frac{\partial}{\partial z} \ln \bar{B} \right). \quad (6.2)$$

It can be easily checked that the second hamiltonian structure for the superfields $B(Z), \bar{B}(Z)$ is rather simple

$$\{B(Z_1) \bar{B}(Z_2)\} = \frac{48}{c} D \bar{D} \delta(Z_1 - Z_2) \quad (6.3)$$

and is related to the $N = 2$ superconformal algebra (3.29) through the transformation (6.2). Thus in the case of the bosonic superfields $B(Z), \bar{B}(Z)$ the transformation (6.2) is defined to be a Poisson map relating the Poisson structure (6.3) and the $N = 2$ superconformal algebra. We would like to stress that the expression (6.2) gives new classical non-polynomial Miura-like realization of $N = 2$ superconformal algebra in terms of free bosonic chiral-anti-chiral superfields $B(Z), \bar{B}(Z)$ with Poisson brackets (6.3). It is very interesting to find the applications of the realization (6.2) to e.g. $N = 2$ superstrings, superconformal field theories, etc.

Let us note that the transformations (6.1), (6.2) are not self-conjugate, but the conjugate ones are also allowed transformations.

To estimate the relation between two different descriptions of $N = 2$ (1, 1)-KdV hierarchy in terms of fermionic $F(Z), \bar{F}(Z)$ (6.1) and bosonic $B(Z), \bar{B}(Z)$ (6.2) superfields one can equate (6.1) and (6.2). The resulting relation is

$$\bar{B} D \bar{F} \exp \left(\frac{1}{2} \partial^{-1} (F \bar{F} + B \bar{B}) \right) = \text{const} . \quad (6.4)$$

6.2 $N=2$ super-Boussinesq case ($a^{(1)} = 20/c$)

This second non trivial example of the $N = 2$ (1, n)-KdV hierarchies at $n = 2$ corresponds to the case of four chiral-anti-chiral superfields with Lax operator (4.2). There are three different possibilities: a) all four superfields are fermionic ($n = 2, m = 0$) b) two superfields are fermionic and two bosonic ($n = 1, m = 1$) and c) all superfields are bosonic ($n = 0, m = 2$). As we showed in the previous section, for the cases a) and c) the bosonic limits coincide with four component GNLS hierarchy. The equivalence between GNLS and (1, n)-KdV hierarchies was

proved in [22]. The simplest $N = 2$ supersymmetric extension of the bosonic transformations from the four component GLNS to the $(1, 2)$ -KdV hierarchies look as follows

$$\begin{aligned}
J &= \frac{c}{24} \left[\frac{1}{2} F_i \bar{F}_i + \frac{\partial}{\partial z} \ln \left(D \bar{F}_2 D \bar{F}_1' - D \bar{F}_2' D \bar{F}_1 + \kappa D \left(F_i \bar{F}_i \right) D \left(\bar{F}_1 \bar{F}_2 \right) \right) \right], \\
T &= -\frac{c}{16} \left[\frac{1}{2} \left(F_i' \bar{F}_i - F_i \bar{F}_i' - \frac{1}{2} (F_i \bar{F}_i)^2 \right) - \frac{16}{c} \left(\frac{5}{2} [D, \bar{D}] J - \frac{20}{c} J^2 \right) \right. \\
&\quad \left. + \frac{\partial}{\partial t_2} \ln \left(D \bar{F}_2 D \bar{F}_1' - D \bar{F}_2' D \bar{F}_1 + \kappa D \left(F_i \bar{F}_i \right) D \left(\bar{F}_1 \bar{F}_2 \right) \right) \right], \tag{6.5}
\end{aligned}$$

where t_2 is related to t in equation (3.4) by $t_2 = -3t$ and time derivatives of $F_i(Z), \bar{F}_j(Z)$ are defined by (4.5). It can be checked that the (6.5) reproduces the correct expressions for the bosonic components of the superfields $J(Z)$ and $T(Z)$ (3.4) in terms of bosonic components of the fermionic superfields (case a) $F_1, \bar{F}_1, F_2, \bar{F}_2$, see the transformations in [22]. But unfortunately (6.5) fails to give the true transformations if the fermionic components are included. This is a bit surprising, since other terms which preserve the $GL(2)$ symmetry of the equations (4.5) for the superfields $F_1, \bar{F}_1, F_2, \bar{F}_2$, and can be added to (6.5), are non-local (the terms in the (6.5) which are proportional to the parameter κ are the only local ones which possess $GL(2)$ symmetry and they disappear in the bosonic limit). As for case b), the bosonic limit is given by the direct sum of two NLS equations and so it cannot be related with the $(1, 2)$ -KdV bosonic hierarchy. Thus the problem of superfield transformations between the systems (4.2) we constructed in this paper and the $N = 2$ super $(1, n)$ -KdV hierarchies with $N = 2$ super W_n algebra second Hamiltonian structure, is still open.

7 Conclusion

In this paper we have presented the results of our study of the possible $N = 2$ supersymmetric integrable hierarchies with $N = 2$ W_n superalgebra as the second hamiltonian structure. We have presented our *Conjecture* about the Lax operators for their bosonic limits. Using it one can reconstruct the supersymmetric hamiltonians. We have also constructed the $N = 2$ supersymmetric extensions of bosonic GNLS hierarchies as well as of $(1, n) \oplus (1, m)$ -KdV hierarchies and their Lax operators in terms of $N = 2$ superfields. We have found a new non-polynomial realization for the $N = 2$ superconformal algebra and proposed a new definition of super-residues for degenerate $N = 2$ supersymmetric pseudodifferential operators. The transformation between different superfield representations has been analyzed for the simplest cases. A detailed analysis of this complicated problem is under way.

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